Uncertainty and entropies of classical channels

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Many results in this presentation are from our published manuscript

Inevitable negativity: Additivity commands negative quantum channel entropy

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What is uncertainty?

Given a random variable X which has n possible values, $\{x_1, x_2, \dots, x_n\}$.

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$$\mathbf{p} = \left(\tfrac{3}{6} \quad \tfrac{1}{6} \quad \tfrac{1}{6} \quad \tfrac{1}{6} \quad 0 \quad 0\right)^T$$

⟵ How can we quantify its uncertainty?

(Shannon) Entropy

$$H(\mathbf{p}) = \sum_{i \in [n]} -p_i \log_2(p_i)$$

- Dependent on the distribution
- Axiomatically defined using information processing task

Variance

$$\sigma^2 = \mathbb{E}\left[X^2 - \mu^2\right]$$

- Dependent on the distribution and the value of X

A family of entropies

α-Rényi entropy ($\alpha \in \mathbb{R}_+$)

$$H_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \log\left(\sum_{x \in [n]} p_x^\alpha \right).$$

Shannon entropy: $H_1(\mathbf{p}) = -\sum_{y \in [n]} p_y \log_2(p_y)$
$\leftrightarrow$ probability of obtaining a long string drawn from an i.i.d. source = $2^{-H_1(\mathbf{p})}$.

Max-entropy: $H_0(\mathbf{p}) = \log_2 |\mathrm{supp}(\mathbf{p})|$
$\leftrightarrow$ number of possibilities = $2^{H_0(\mathbf{p})}$.

Min-entropy: $H_\infty(\mathbf{p}) = -\log_2(\max_{y \in [n]} p_y)$
$\leftrightarrow$ probability of giving a correct guess of an outcome = $2^{-H_\infty(\mathbf{p})}$

Denoted by $\mathrm{supp}(\mathbf{p}) = \{ y : p_y \neq 0 \}$ is a set of outcomes y with non-zero probability.

Entropy and uncertainty

Shannon entropy

$$H(\mathbf{p}) = -\sum_{x\in\mathcal{I}} p_x \log p_x.$$

α-Rényi entropy ($\alpha \in \mathbb{R}_+$)

$$H_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \log\left(\sum_{x\in[n]} p_x^\alpha\right).$$

The least uncertained distribution: If $\mathbf{p} = \mathbf{e}_1 = (1, 0, \ldots, 0)^T$, knowing the value of X with certainty,

$$H(\mathbf{e}_1) = H_\alpha(\mathbf{e}_1) = 0.$$ ← the *least* entropy can be.

The most uncertained distribution: If $\mathbf{p} = \mathbf{u}^{(n)} = \frac{1}{n}(1, 1, \ldots, 1)^T$, every outcome is equally likely,

$$H(\mathbf{u}^{(n)}) = H_\alpha(\mathbf{u}^{(n)}) = \log(n).$$ ← the *most* entropy can be.

Nice common properties: non-negative, additive, invariant with permutation and adding zero.

Classical states

- A die rolled a five,

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$$\mathbf{p} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T$$

- A fair die is tossed,

$$\mathbf{p} = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}^T.$$

Statics

Classical channels

- sending a bit through a telephone line,

$$\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathcal{N} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$$

- tossing a fair die,

$$\mathcal{P}_{\mathbf{p}} = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}^T.$$

Dynamics

Question: can we have entropy for classical channels as well?

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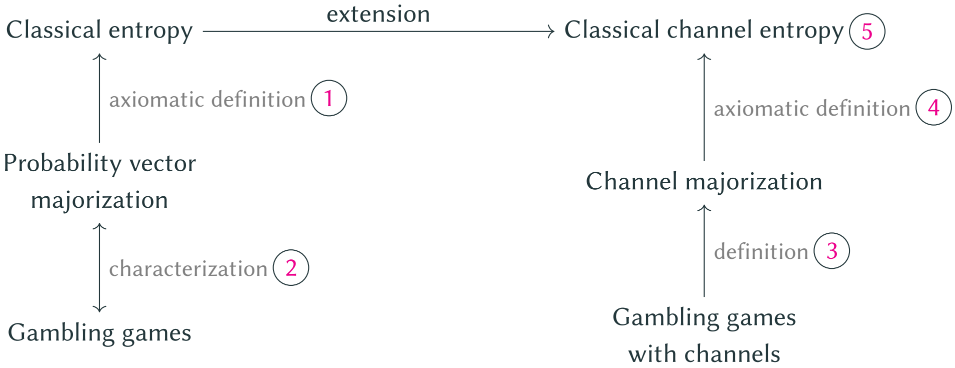

Axiomatic definition of classical state entropy

Def. An entropy is a function H : ProbabilityVectors $\to \mathbb{R}_+$ such that

1. it is an antitone under majorization relation
2. it is additive under a (Kronecker) tensor product

Majorization ($\succ$) is a preorder on the set of probability vectors.
Def. $\mathbf{p} \succ \mathbf{q}$ if for any $k \in [n]$ the sum of k largest elements of $\mathbf{p}$ is larger or equal to that of $\mathbf{q}$.

$$\mathbf{p} \succ \mathbf{q}$$

$$\mathbf{p} \text{ is more certain than } \mathbf{q} \underbrace{\implies H(\mathbf{p}) \leq H(\mathbf{q})}_{\text{being antitone}}$$

Additivity under tensor product: two independent random variables $X \sim \mathbf{p}$, $Y \sim \mathbf{q}$ has a joint probability vector $\mathbf{p} \otimes \mathbf{q}$. Additivity means

$$H(\mathbf{p} \otimes \mathbf{q}) = H(\mathbf{p}) + H(\mathbf{q})$$

Majorization and uncertainty

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$$\mathbf{p} = (\tfrac{3}{6} \quad \tfrac{1}{6} \quad \tfrac{1}{6} \quad \tfrac{1}{6} \quad 0 \quad 0)^T$$

k-**game**: within k guesses, correctly guess the value of a random variable $X \sim \mathbf{p}$.

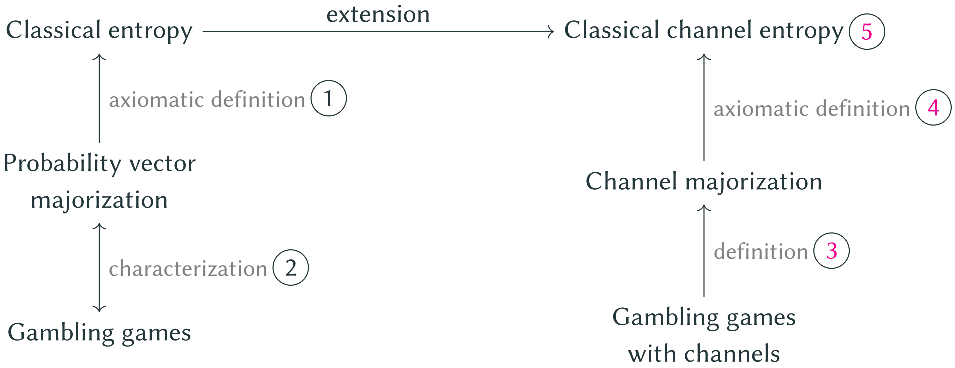
> $\mathbf{p} \succ \mathbf{q}$ is equivalent to $\mathbf{p}$ is at more or equally likely to win any k-game than with $\mathbf{q}$.

Examples,

$$(1,0,0)^T \succ (\frac{2}{3}, \frac{1}{3}, 0)^T \succ (\frac{1}{2}, \frac{1}{2}, 0)^T \succ \frac{1}{3}(1,1,1)^T.$$

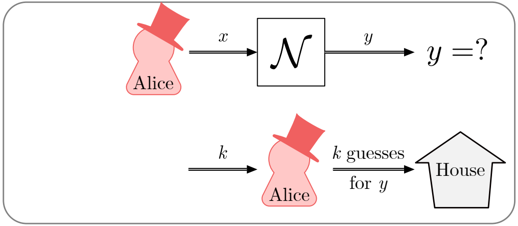
$(1,0,0)^T$ is called the maximal element as it *majorizes* everything.
$\frac{1}{3}(1,1,1)^T$ is called the minimal element as it *is majorized* by all vector in $\mathrm{Prob}(3)$.

We're going to discuss new results!



Extending game to classical channel

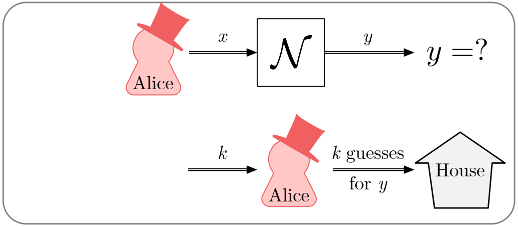
Goal: guess the value of the output of a channel within k guesses.
The player can pick any input x but k is randomly picked.



The player knows prior to giving input x

- transition matrix N associated with the channel $\mathcal{N}$ and
- Probability to have k number of guesses.

Goal: guess the value of the output of a channel within k guesses.

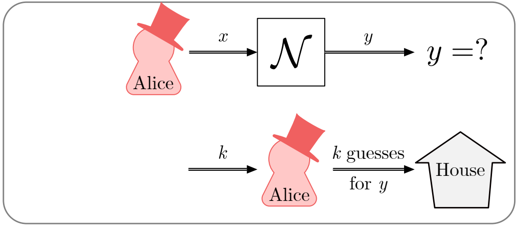


$$N = \begin{bmatrix} 0.7 & 0.4 & \leftarrow a \\ 0.15 & 0.1 & \leftarrow b \\ 0.15 & 0.5 & \leftarrow c \end{bmatrix}$$
$$\uparrow \quad \uparrow$$
$$x = 1, \quad 2$$

k	Pick x	Guess y	Winning Rate
1	1	a	70%
2	2	a and c	90%
1 (50%), 2 (50%)	1	a and b	77.5%

Majorization of classical channels

Goal: guess the value of the output of a channel within k guesses.
The player can pick any input x but k is randomly picked.

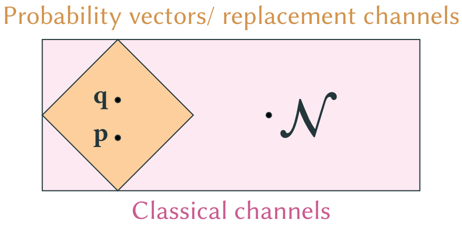


Suppose $\mathcal{N}^{X \to Y}$ and $\mathcal{M}^{X' \to Y}$ two classical channels. $\mathcal{N} \succ \mathcal{M}$ if for any distribution of k, the winning chance with channel $\mathcal{N}$ is higher.

Extending majorization to channels

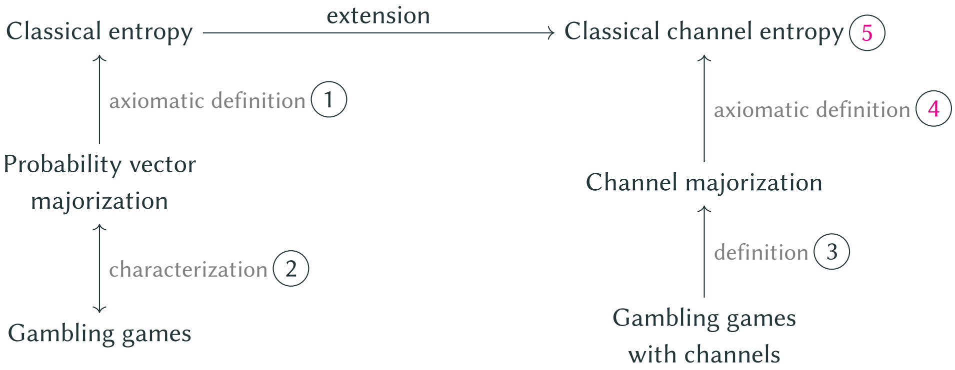
This extended relation

1. defines uncertainty inherent in classical channel,
2. reduces to probability vector majorization on replacement channel,



3. has an identity channel as a maximal element, and
4. has a maximally randomizing channel as a minimal element.

Question: can we have entropy for classical channels as well?



Channel entropy

Definition. An entropy is a function H : ClassicalChannel $\to \mathbb{R}_+$ such that

1. it is an antitone under Channel majorization relation,

$$\mathcal{N} \succ \mathcal{M} \implies H(\mathcal{N}) \leq H(\mathcal{M}),$$

2. it is additive under a tensor product,

$$H(\mathcal{N} \otimes \mathcal{M}) = H(\mathcal{N}) + H(\mathcal{M}).$$

Existence?

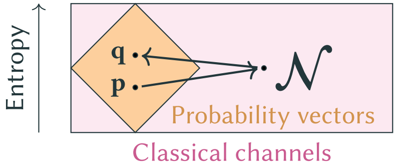
Optimal extensions of an antitone

Maximal extension: entropy of the least noisy probability vector that is still more noisy than the channel.

$$\overline{\mathbb{H}}(\mathcal{N}) = \inf_{\substack{\mathbf{q}\in\mathrm{Prob}(m)\\ m\in\mathbb{N}}} \left\{ \mathbb{H}(\mathbf{q}) \,:\, \mathcal{N} \succ \mathbf{q} \right\}$$

Minimal extension: entropy of the most noisy probability vector that is still less noisy than the channel.

$$\underline{\mathbb{H}}(\mathcal{N}) = \sup_{\substack{\mathbf{p}\in\mathrm{Prob}(m)\\ m\in\mathbb{N}}} \left\{ \mathbb{H}(\mathbf{p}) \,:\, \mathbf{p} \succ \mathcal{N} \right\}$$



Gour, G. and Tomamichel, M. "Optimal extensions of resource measures and their applications". In: Phys. Rev. A 102 6 062401 (2020)

Examples of classical channel entropy

The maximal extension of a Rényi entropy H_α is additive and it is the *min-entropy output*,

$$\overline{\mathbb{H}}(\mathcal{N}) = \min_{y\in[m]} \mathbb{H}(\mathbf{p}_y) \tag{1}$$

where $\mathbf{p}_y \stackrel{\text{def}}{=} \mathcal{N}(\mathbf{e}_y)$.

α-Rényi entropies

$$H_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \log\left(\sum_{y\in[n]} p_y^\alpha\right).$$

Denoted by $\operatorname{supp}(\mathbf{p}) = \{y : p_y \neq 0\}$ is a set of outcomes y with non-zero probability.

Unique entropy extensions

$$\text{Max-entropy} \to H_0(\mathbf{p}) = \log|\operatorname{supp}(\mathbf{p})|,$$

$$\text{Shannon entropy} \to H_1(\mathbf{p}) = -\sum_{y\in[n]} p_y \log(p_y),$$

$$\text{Min-entropy} \to H_\infty(\mathbf{p}) = -\log\left(\max_{y\in[n]} p_y\right).$$

Nonexample: entropy of the Choi state

Suppose that $\mathcal{N} \in \mathrm{CPTP}(X \to Y)$ and $\mathfrak{J}_{\mathcal{N}} \in \mathfrak{L}(XY)$ is its Choi matrix and $\hat{\mathfrak{J}}_{\mathcal{N}}$ to be the normalized Choi matrix. A function f of $\mathcal{N}$ is defined by

$$f(\mathcal{N}) = H\left(\hat{\mathfrak{J}}_{\mathcal{N}}\right) - \log(|X|)$$

where H is a von Neumann entropy. The function f is purposed to be an entropy function[1,2].

Choi state

In probability vector representation,

$$\hat{\mathfrak{J}}_{\mathcal{N}} = \sum_{x \in [n]} \frac{1}{n} \mathbf{e}_x \otimes \mathcal{N}(\mathbf{e}_x).$$

A classical state that correlate each choice of input with its output.

f is not a channel entropy

$\mathbf{e}_1 \sim (\mathbf{e}_1, \mathbf{p})$ for any probability vector $\mathbf{p}$.
If H is a channel entropy, then

$$H(\mathbf{e}_1) = H(\mathbf{e}_1, \mathbf{p}).$$

However, there is $\mathbf{p}$ such that $f(\mathbf{e}_1) \neq f(\mathbf{e}_1, \mathbf{p})$.

[1] J. Czartowski, D. Braun, and K. Życzkowski. "Trade-off relations for operation entropy of complementary quantum channels". Int. J. Quantum Inf. 17 05 1950046 (2019).
[2] Y. Chu et al. "An entropy function of a quantum channel". Quantum Inf. Process. 22 1 27 (2022)

Conclusion

We defined entropy of a classical channel axiomatically and concretely.

Axiomatically:

1. Axiomatic definition of classical state entropy and majorization.
2. Extension of majorization to classical channels.
3. Axiomatic definition of classical channel entropy follows from extended majorization.

Concretely:

1. Antitone of the extended majorization can be extended from a state entropy.
2. An extension of α-Rényi entropy to a channel entropy.